Special quadratic first integrals of geodesics

1971 J. Phys. A: Gen. Phys. 4 756

(http://iopscience.iop.org/0022-3689/4/6/002)

J. Phys. A: Gen. Phys., 1971, Vol. 4. Printed in Great Britain

Special quadratic first integrals of geodesics

C. D. COLLINSON
Department of Applied Mathematics, University of Hull, Hull, Yorks., UK
MS. received 22*nd March* 1971

Abstract. The first set of integrability conditions for the existence of a quadratic first integral of the geodesic equations in a general Riemannian space are obtained. A class of special quadratic first integrals is defined and their properties discussed.

1. Introduction

Several papers have appeared recently investigating the existence of quadratic first integrals of the geodesic equations in a Riemannian space. The problem is of interest in its own right but is also motivated since knowledge of the geodesics is of help in analysing given space-times in general relativity (Walker and Penrose 1970).

The differential equations of the geodesics in a Riemannian space V_n are

$$\frac{d^2x^i}{d\lambda^2}+\Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0.$$

These equations will admit a quadratic first integral

$$a_{ij}\dot{x}^i\dot{x}^j = \text{constant} \tag{1.1}$$

provided that the symmetric tensor a_{ij} satisfies the equation (Eisenhart 1949, p. 128)

$$a_{(ij;k)} = 0 \tag{1.2}$$

where the round brackets denote symmetrization. This equation is a generalization of Killing's equation

$$a_{(i;j)} = 0 \tag{1.3}$$

which is the condition for the existence of a linear first integral

$$a_i\dot{x}^i = \text{constant}.$$

A symmetric tensor satisfying equation (1.2) is called a Killing tensor of rank two. Thomas (1946) has shown that the maximal number of Killing tensors admitted by a given space V_n is $n(n+1)^2(n+2)/12$.

The integrability conditions for equation (1.2) have not appeared in the literature; they are found in § 2. These integrability conditions involve second derivatives of the curvature tensor and seem far too complex to analyse in practice. For this reason it is appropriate to investigate whether any class of special quadratic first integrals can be defined in a natural manner. One such class consists of those first integrals generated by a covariantly constant tensor (Levine and Katzin 1968). Another class is defined in § 2, the first integrals of this class being analogues of those first integrals of a flat space which depend linearly on the (cartesian) coordinates. The integrability conditions for the existence of these special quadratic first integrals are obtained in § 3; their form bears a close resemblance to the integrability conditions for the existence of a Killing vector.

In § 4 it is shown that the special quadratic first integrals can be written as the sum of products of two linear first integrals only if the space admits a covariantly constant vector. This condition is very restrictive and so it is likely that the search for special quadratic first integrals might prove useful in the analysis of the geodesics of given relativistic space–times possessing groups of motion of order less than or equal to two. Several results relevant to such space–times are given in § 5.

2. The integrability conditions for general quadratic first integrals

In what follows the curvature tensor R_{ijkl} is defined by the Ricci identity

$$a_{ij;kl} - a_{ij;lk} = R^p{}_{ikl} a_{pj} + R^p{}_{jkl} a_{ip}. \tag{2.1}$$

Suppose a_{ij} is a Killing tensor; that is a_{ij} is a symmetric tensor satisfying equation (1.2). Let u^i, w^i be two arbitrary vectors. Then, using equations (1.2) and (2.1)

$$\begin{aligned} a_{ij;kl} v^i v^j w^k w^l &= -2a_{jk;il} v^i v^j w^k w^l \\ &= (-2a_{jk;li} - 2R^p{}_{jil} a_{pk} - 2R^p{}_{kil} a_{jp}) v^i v^j w^k w^l \\ &= (a_{kl;ji} - 2R^p{}_{jil} a_{pk} - 2R^p{}_{kil} a_{jp}) v^i v^j w^k w^l. \end{aligned}$$

Hence

$$a_{ij;(kl)} = a_{kl;(ij)} - 2R^p{}_{(ji)(l} a_{k)p} + 2R^p{}_{(kl)(i} a_{j)p}. \tag{2.2}$$

Using (2.1) the identity

$$a_{ij;kl} - a_{kl;ji} + a_{ij;lk} - a_{kl;ij} = 2a_{ij;(kl)} - 2a_{kl;(ij)}$$

can be rewritten as

$$\begin{aligned} 2a_{ij;kl} - 2a_{kl;ij} = 2a_{ij;(kl)} - 2a_{kl;(ij)} &+ R^p{}_{ikl} a_{pj} + R^p{}_{jkl} a_{ip} \\ &- R^p{}_{kij} a_{pl} - R^p{}_{lij} a_{kp}. \end{aligned}$$

Substituting (2.2) into this yields

$$a_{ij;kl} - a_{kl;ij} = 2R^p{}_{lk(i} a_{j)p} - 2R^p{}_{ji(k} a_{l)p}. \tag{2.3}$$

Differentiating (2.3) and making use of the Ricci identity gives

$$a_{ij;klm} = a_{kl;m(ij)} + T_{(ij)klm} \tag{2.4}$$

where

$$\begin{aligned} T_{ijklm} = &-R^p{}_{ijk} a_{lp;m} - R^p{}_{ijl} a_{kp;m} + R^p{}_{ijm} a_{kl;p} \\ &+ 2R^p{}_{kjm} a_{pl;i} + 2R^p{}_{ljm} a_{pk;i} + 2R^p{}_{lkj} a_{ip;m} \\ &+ 2R^p{}_{lkj;m} a_{ip} + (R^p{}_{klm;j} - R^p{}_{jik;m}) a_{pl} + (R^p{}_{lim;j} - R^p{}_{jil;m}) a_{pk}. \end{aligned} \tag{2.5}$$

It follows from (2.4) and (1.2) that

$$a_{ij;klm} + a_{ij;lmk} + a_{ij;mkl} = T_{(ij)klm} + T_{(ij)lmk} + T_{(ij)mkl}.$$

Using the Ricci identity to permute the derivative indices on the second two terms on the left-hand side yields the equation

$$3a_{ij;klm} = T_{(ij)klm} + T_{(ij)lmk} + T_{(ij)mkl} + S_{(ij)klm} \tag{2.6}$$

where

$$\begin{aligned} S_{ijklm} = &\, 4R^p{}_{ikm} a_{pj;l} + 2R^p{}_{ikl} a_{pj;m} + 2R^p{}_{ilm} a_{pj;k} \\ &+ R^p{}_{lkm} a_{ij;p} + R^p{}_{klm} a_{ij;p} + 2R^p{}_{ikl;m} a_{pj} + 2R^p{}_{ikm;l} a_{pj}. \end{aligned} \tag{2.7}$$

It follows that the third and higher derivatives of a_{ij} can all be found, at a point, in terms of a_{ij} and its first two derivatives. The first set of integrability conditions can now be derived by equating the expressions for $a_{ij;klmn} - a_{ij;klnm}$ obtained from (2.6) and from an application of the Ricci identity. These integrability conditions involve first and second derivatives of the curvature tensor and so are too complex to be of practical use. However if the V_n is assumed to be a space of constant curvature it is easily shown that the integrability conditions are identically satisfied. Hence a space of constant curvature admits the maximal number of quadratic first integrals. This result has been proved, less directly, by Levine and Katzin (1965).

Equation (2.3) suggests the definition of a class of quadratic first integrals, called here 'special' quadratic first integrals, by the requirement that the tensor a_{ij} satisfies equation (1.2) and the supplementary equation

$$a_{ij;kl} = +2R^{p}{}_{ik(i}a_{j)p}. \tag{2.8}$$

Such special quadratic first integrals clearly correspond to those first integrals, in a flat space-time, which depend linearly upon the (cartesian) coordinates.

3. The integrability conditions for special quadratic first integrals

The required integrability conditions are found by equating the expressions obtained for $a_{ij;klm} - a_{ij;kml}$ from equation (2.8) and from the Ricci identity. Using (1.2), and relabelling indices, the integrability conditions can be written as

$$R_{ijk(l;}{}^{p}a_{m)p} + R_{pjk(l}a_{m)}{}^{p}{}_{;i} + R_{ipk(l}a_{m)}{}^{p}{}_{;j} + R_{ijp(l}a_{m)}{}^{p}{}_{;k} + R_{ijkp}a^{p}{}_{(m;l)} = 0. \tag{3.1}$$

The form of equation (3.1) is very similar to the form of the integrability conditions for Killing's equation (1.3), namely

$$R_{ijkl;}{}^{p}a_{p} + R_{pjkl}a^{p}{}_{;i} + R_{ipkl}a^{p}{}_{;j} + R_{ijpl}a^{p}{}_{;k} + R_{ijkp}a^{p}{}_{;l} = 0. \tag{3.2}$$

This is hardly surprising since a Killing vector necessarily satisfies an equation analogous to (2.8), namely

$$a_{i;jk} = R^{p}{}_{kji}a_{p}. \tag{3.3}$$

For a space of nonzero constant curvature the integrability conditions (3.1) yield, on contraction, $a_{ij;k} = 0$. Substituting this into (2.8) then gives $a_{ij} = 0$ and so spaces of constant curvature do not admit special quadratic first integrals. The maximal number of special quadratic first integrals is $n(n+1)(2n+1)/6$ and this number is admitted by a flat space.

4. A theorem on reducible special quadratic first integrals

Suppose a special quadratic first integral can be written as the sum of products of two linear first integrals. Then

$$a_{ij} = \sum_{P,Q=1}^{2} C^{PQ}a_{Pi}a_{Qj} \tag{4.1}$$

where the two vectors a_{1i}, a_{2i} satisfy Killing's equation (1.3) and the coefficients $C^{PQ} = C^{QP}$ are constants. Substituting (4.1) into (2.8) and making use of (3.3) yields

$$\sum_{P,Q=1}^{2} C^{PQ}[a_{Pi;k}a_{Qj;l} + a_{Pi;l}a_{Qj;k}] = 0. \tag{4.2}$$

Multiplying (4.2) by $v^k v^l$, with v^k arbitrary, gives

$$C^{11}b_{1i}b_{1j}+C^{12}(b_{1i}b_{2j}+b_{1j}b_{2i})+C^{22}b_{2i}b_{2j} = 0 \tag{4.3}$$

where $b_{Pi} = a_{Pi;k}v^k$. Now suppose that $C^{11} \neq 0$. Then either $b_{1i}+C^{12}b_{2i}/C^{11} = 0$ or $b_{1i}+C^{12}b_{2i}/C^{11} = \alpha b_{2i}$. Substituting this last equation into (4.3) yields

$$[(C^{11}\alpha)^2+C^{11}C^{22}-(C^{12})^2]b_{2i}b_{2j} = 0.$$

Hence either $b_{2i} = 0$ or α can be found in terms of C^{PQ}. It follows that b_{1i} and b_{2i} are linearly dependent

$$\sum_{P=1}^{2} K^P b_{Pi} = 0 \tag{4.4}$$

the coefficients K^P being constants constructed from the C^{PQ}. The same conclusion is arrived at for the case $C^{11} = 0$. Since v^k was chosen arbitrarily it follows from (4.4) that

$$\sum_{P=1}^{2} K^P a_{Pi;k} = 0$$

or

$$(K^1 a_{1i}+K^2 a_{2i})_{;k} = 0.$$

The results can be stated as the *theorem*: A special quadratic first integral in a space V_n can be written in terms of two linear first integrals only if the space admits a covariantly constant vector field.

If the special quadratic first integral is assumed to be the sum of products of three linear first integrals then it can again be shown that the vectors b_{Pi} ($P = 1, 2, 3$) are linearly dependent but now the coefficients K^P are no longer necessarily constants.

5. Special quadratic first integrals in empty space–times

In an empty space–time the Ricci tensor R_{ij} vanishes and so contracting (3.1) on j,k yields

$$R_{ipk(l}a_{m)}{}^{p;k}+R_{ikp(l}a_{m)}{}^{p;k} = 0.$$

Using (1.2) this equation can be written as

$$R_{ipk(l}a^{pk}{}_{;m)} = 0. \tag{5.1}$$

Adding the two equations obtained by cyclically permuting the indices *ilm* on (5.1) and subtracting (5.1) itself yields

$$R_{ipkm}a^{pk}{}_{;l} = 0. \tag{5.2}$$

This equation imposes certain restrictions. Of particular interest is the case of space–times of Petrov type D. Using the notation of Newman and Penrose (1962) a null tetrad can be chosen so that the only nonzero tetrad component of the curvature tensor is ψ_2. The tetrad form of (5.2) then yields $a_{ij;k} = 0$, provided that $\psi_2+\bar{\psi}_2 \neq 0$. The condition $\psi_2+\bar{\psi}_2 \neq 0$ is equivalent to the condition that the geodesic rays should *not* be twist free. Substituting $a_{ij;k} = 0$ into equation (2.8) then gives

$$R^p{}_{lk(i}a_{j)p} = 0$$

and the tetrad form of this equation leads to $a_{ij} = 0$. Hence the *theorem*: Empty space–times of Petrov type D possessing twisting geodesic rays (the Kerr solution) cannot admit a special quadratic first integral.

It follows that the quadratic first integral found by Walker and Penrose (1970) cannot be special.

It can also be shown that empty space–times of Petrov type N possessing twisting geodesic rays cannot admit a special quadratic first integral. The proof, however, involves analysis of the full integrability conditions and so will not be given.

References

EISENHART, L. P., 1949, *Riemannian Geometry* (Princeton: Princeton University Press).
LEVINE, J., and KATZIN, G. H., 1965, *Tensor N.S.*, **16**, 97–104.
—— 1968, *Tensor N.S.*, **19**, 113–20.
NEWMAN, E. T., and PENROSE, R., 1962, *J. math. Phys.*, **3**, 565–78.
THOMAS, T. Y., 1946, *Proc. Natn. Acad. Sci.*, **32**, 10–5.
WALKER, M., and PENROSE, R., 1970, *Commun. math. Phys.*, **18**, 265–74.